Fecundity regulation in a spatial birth-and-death process

Based on a joint paper with

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Description of process

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- Henceforth, $|\eta|$ denotes the number of points in a discrete finite set $\eta \subset \mathbb{R}^{\mathrm{d}}$.
- We identify a configuration $\eta \in \Gamma$ with a discrete (counting) measure on $(\mathbb{R}^{\mathrm{d}}, \mathscr{B}_{\mathrm{b}}(\mathbb{R}^{\mathrm{d}}))$ defined by assigning a unit mass to each atom at $x \in \eta$:

$$\eta \leftrightarrow \sum_{x \in \eta} \delta_x$$

Basic objects

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- We denote then

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- We define a sequential topology on Γ_G by assuming that $\eta_n \to \eta$, $n \to \infty$, if only

$$\lim_{n\to\infty} \langle f, \eta_n \rangle = \langle f, \eta \rangle$$

  for all $f \in C_b(\mathbb{R}^{\mathrm{d}})$ (the space of bounded continuous functions on $\mathbb{R}^{\mathrm{d}}$) such that $|f(x)| \le M_f G(x)$ for some $M_f > 0$ and all $x \in \mathbb{R}^{\mathrm{d}}$.

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- Let $\mathscr{B}_G(\Gamma)$ denote the corresponding Borel σ-algebra.

Definition

Let $b : \mathbb{R}^{\mathrm{d}} \times \Gamma_G \to \mathbb{R}_+ := [0, \infty)$ *be a measurable function. We describe a spatial birth-and-death process* $\eta : \mathbb{R}_+ \to \Gamma_G$ *with the unit death rate and the birth rate* b *through the following three properties:*

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1. *If the system is in a state* $\eta_t \in \Gamma_G$ *at the time* $t \in \mathbb{R}_+$*, then the probability that a new particle appears (a "birth" happens) in a* $B \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^{\mathrm{d}})$ *during a time interval* $[t, t + \Delta t]$ *is*

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3. *With probability* 1 *no two events described above happen simultaneously.*

Definition

- *Let* $\widetilde{N}$ *be the Poisson point process on* $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+^2$ *with mean measure* $ds \times dx \times du \times e^{-r}dr$. *The process* $\widetilde{N}$ *is said to be compatible w.r.t. a filtration* $\{\mathcal{F}_t\}$ *if, for any measurable* $A \subset \mathbb{R}^d \times \mathbb{R}_+^2$, $\widetilde{N}([0,t],A)$ *is* $\mathcal{F}_t$*-measurable and* $\widetilde{N}((t,s],A)$ *is independent of* $\mathcal{F}_t$ *for* $0 < t < s$.

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- *Let* η_0 *be a* Γ_G*-valued* $\mathcal{F}_0$*-measurable random variable independent on* $\widetilde{N}$. *Consider a point process* $\widetilde{\eta}_0$ *on* $\mathbb{R}^d \times \mathbb{R}_+$ *obtained by attaching to each point of* η_0 *an independent unit exponential random variable. Namely, if* $\eta_0 = \{x_i : i \in \mathbb{N}\}$ *then* $\widetilde{\eta}_0 = \{(x_i, \tau_i) : i \in \mathbb{N}\}$ *and* $\{\tau_i\}$ *are independent unit exponentials, independent of* η_0 *and* $\widetilde{N}$.

Definition

- *We will say that a process* $(\eta_t)_{t\geq 0}$ *with sample paths in the Skorokhod space* $D_{\Gamma_G}[0,\infty)$ *has the unit death rate and the birth rate* b *if it is adapted to a filtration* $\{\mathcal{F}_t\}$ *w.r.t. to which* $\widetilde{N}$ *is compatible and if, for any* $B \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^{\mathrm{d}})$*, the following equality holds almost surely*

$$\eta_t(B) = \int\limits_{(0,t]\times B\times\mathbb{R}_+^2} I_{[0,b(x,\eta_{s-})]}(u) I_{\{r>t-s\}} \widetilde{N}(ds,dx,du,dr) + \int\limits_{B\times\mathbb{R}_+} I_{\{r>t\}} \widetilde{\eta}_0(dx,dr) \tag{1}$$

where $\eta_t(B) = |\eta_t \cap B|$ *is the number of points in* B*; note that henceforth we use configurations and counting measures interchangeably.*

Generator

The (heuristic) generator of our process is

$$LF(\eta) = \sum_{x \in \eta} \Big(F(\eta \setminus x) - F(\eta) \Big) + \int_{\mathbb{R}^d} b(x, \eta) \Big(F(\eta \cup x) - F(\eta) \Big) dx.$$

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Example: nonlocal branching process (see e.g. Durrett'1979)

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Alternative interpretation: after an exponential time with the unit rate each particle at y dies and produces either 0 off-springs with porbability $\frac{1}{1+\langle a\rangle}$ or two off-springs with probability $\frac{\langle a\rangle}{1+\langle a\rangle}$, where $\langle a\rangle = \int_{\mathbb{R}^d} a(x)\,dx$, so that on off-springs stays at the prarent's position and the other is at x distributed according to the kernel $\langle a\rangle^{-1}a(x-y)$.

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$$\mathbb{E}\eta_t(B) \sim e^{(\langle a\rangle-1)t}\mathrm{vol}(B).$$

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How to regulate?

Theorem (corollary of results in Garcia/Kurtz'2006)

Suppose that

$$\mathbf{b} := \sup_{\substack{x\in\mathbb{R}^d \\ \eta\in\Gamma_G}} b(x,\eta) < \infty,$$

and, for some $M > 0$*,*

$$\sup_{\eta\in\Gamma_G} \left| b(x,\eta\cup y) - b(x,\eta) \right| \le MG(x-y), \quad x,y\in\mathbb{R}^d.$$

Then there exists a unique solution to (1)*.*

If, additionally, both b *and* η_0 *are translation invariant, then* η_t *is translation invariant for* $t > 0$*.*

Density dependent fecundity

Let $0 \le a, c, \varphi \in L^1(\mathbb{R}^{\mathrm{d}})$. Consider

$$b(x,\eta) = \sum_{y\in\eta} a(x-y)\left(1 + \sum_{z\in\eta\setminus\{y\}} c(z-y)\right)\exp\left(-\sum_{z\in\eta\setminus\{y\}} \varphi(z-y)\right). \qquad (2)$$

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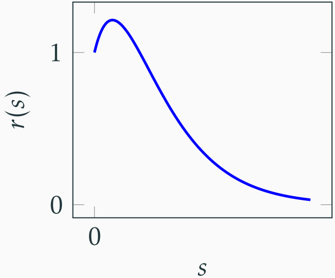
Ecological interest: weak Allee effect, when $p > 1$. In this case the function $r(s) := (1+ps)e^{-s}$ is unimodal:



Lemma

Let $\alpha, \rho > 0$, and let $b_f : \mathbb{R}_+ \to (0,\infty)$ be a bounded decreasing to 0 on $\mathbb{R}_+$ function, such that

$$\int_{\mathbb{R}_+} b_f(s) s^{\mathrm{d}-1}\, ds < \infty.$$

Let $f, g : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}_+$ be measurable functions, f is bounded, such that

$$f(x) \le b_f(|x|), \qquad x \in \mathbb{R}^{\mathrm{d}},$$
$$g(x) \ge \alpha, \qquad |x| \le \rho.$$

Then

$$\sup_{\substack{x\in\mathbb{R}^{\mathrm{d}} \\ \eta\in\Gamma}} \sum_{y\in\eta} f(x-y) \exp\Big\{-\sum_{z\in\eta\setminus y} g(z-y)\Big\} < \infty.$$

Condition 1. There exists $B \geq 1$ and $p \geq 0$, such that, for a.a. $x \in \mathbb{R}^{\mathrm{d}}$,

$$a(x) \leq BG^2(x), \qquad \varphi(x) \leq BG(x), \qquad c(x) \leq p\varphi(x).$$

Condition 2. The function φ is separated from 0 in a neighborhood of the origin.

Theorem

Let b be the birth rate given by (2) and let Conditions 1–2 hold. Then there exists a unique solution to (1).

If, additionally, both b and η_0 are translation invariant, then η_t is translation invariant for $t > 0$.

Properties of processes with bounded birth rates

Examples

By the Lemma above, the fecundity birth rate (2) is globally bounded. Consider two other examples of bounded birth rates known in literature.

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Glauber dynamics in continuum Consider the rate

$$b_{z,\phi}(x,\eta) = z\exp\Bigg(-\sum_{y\in\eta\setminus\{x\}}\phi(x-y)\Bigg),$$

where $z > 0$ and $\phi : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}_+$ is such that $\phi(x) \le BG(x)$, $x \in \mathbb{R}^{\mathrm{d}}$ for some $B > 0$.

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Establishment rate Consider the rate, cf. (2):

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where $0\le a,c,\phi\in L^1(\mathbb{R}^{\mathrm{d}})$ are such that, for $x\in\mathbb{R}^{\mathrm{d}}$,

$$a(x)\le q\phi(x),\qquad c(x)\le p\phi(x),\qquad \phi(x)\le BG(x),$$

for some $q,B>0$, $p\ge 0$.

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1) Consider the so-called **Surgailis process** $(\xi_t)_{t\geq 0}$ with the death rate 1 and the birth rate $\mathbf{b}$, such that $\xi_0 = \eta_0$ a.s. It satisfies the equation

$$\xi_t(B) = \int\limits_{(0,t]\times B\times\mathbb{R}_+^2} I_{[0,\mathbf{b}]}(u) I_{\{r>t-s\}} \widetilde{N}(ds,dx,du,dr) + \int\limits_{B\times\mathbb{R}_+} I_{\{r>t\}} \widetilde{\eta}_0(dx,dr).$$

Auxiliary processes

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Then

$$\eta_t \subset \xi_t \quad \text{a.s.}, t > 0.$$

Corollary

There exists $C > 0$ such that, for each $B \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^{\mathrm{d}})$,

$$\mathbb{E}|\eta_t \cap B| \leq C \,\mathrm{vol}(B) \tag{3}$$

for all $t > 0$, provided that (3) holds for $t = 0$.

2) Consider the Poisson Process $(\Pi_t)_{t \geq 0}$ defined by

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Then

$$\eta_t \setminus \eta_0 \subset \xi_t \setminus \eta_0 \subset \Pi_t, \quad t \geq 0,$$
$$\Pi_s \subset \Pi_t, \quad 0 \leq s \leq t.$$

We consider

$$W(\eta) = \sum_{x\in\eta} h(x) + \sum_{\{x,y\}\subset\eta} \phi(x)\phi(y)K(|x-y|) \in [0,\infty].$$

Here $\phi, h : \mathbb{R}^{\mathrm{d}} \to (0,\infty)$ are separated from 0 on each compact subset of $\mathbb{R}^{\mathrm{d}}$, and $K : (0,\infty) \to (0,\infty)$ is such that

$$\lim_{q\to 0+} K(q) = \infty,$$

$$\int_r^\infty K(q)q^{\mathrm{d}-1}dq < \infty, \quad r > 0,$$

$$C := \sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} \phi(y)K(|x-y|)dy < \infty,$$

$$2C\mathbf{b}\phi(x) \le h(x) \le G(x), \quad x \in \mathbb{R}^{\mathrm{d}}.$$

Proposition

Suppose that $\mathbb{E}W(\eta_0) < \infty$. *Then, for all* $t > 0$,

$$\mathbb{E}W(\eta_t) \leq (1+t)\mathbb{E}W(\eta_0) + \mathbb{E}W(\Pi_t) < \infty$$

Properties

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Corollary

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$$\mathbb{E}\int_0^t \left|LW(\eta_{s-})\right| ds < \infty, \quad t \geq 0.$$

Martingale property

Proposition

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Lemma

The function W *is a Foster–Lyapunov-type function:*

$$LW(\eta) \le \boldsymbol{b}\langle h\rangle - \frac{1}{2}W(\eta), \quad \eta \in \Theta.$$

Theorem

Suppose that $\mathbb{E}W(\eta_0) < \infty$. *Then*

$$\limsup_{t\to\infty} \mathbb{E}W(\eta_t) \leq 2\boldsymbol{b}\langle h \rangle.$$

Let $\delta > 0$ be a small number. For $K > 0$, let τ_K be the return times to the set $\{\zeta \in \Theta : W(\zeta) < K\}$, namely,

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Suppose that $\mathbb{E}W(\eta_0) < \infty$. *Then, for each* $K > \boldsymbol{b}\langle h\rangle$,

$$\mathbb{E}\tau_K \le \frac{\mathbb{E}W(\eta_0)}{K - \boldsymbol{b}\langle h\rangle}.$$

Return times

Let $\delta > 0$ be a small number. For $K > 0$, let τ_K be the return times to the set $\{\zeta \in \Theta : W(\zeta) < K\}$, namely,

$$\tau_K = \inf\Big\{t > \delta \,\Big|\, W(\eta_t) < K\Big\}$$

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Theorem

Assume that $\mathbb{E}W(\eta_0) < \infty$. Then, for all $\theta \in (0, \frac{1}{2})$, there exists $K_\theta > 0$ such that, for $K > K_\theta$,

$$\mathbb{E}e^{\theta\tau_K} < \infty.$$

Uniqueness of the degenerate invariant distribution for sublinear birth rate

The fecundity birth rate (2) satisfies

$$b(x,\eta) \le \sum_{y\in\eta} g(x-y), \quad \eta \in \Gamma_G,\ x \notin \eta, \tag{4}$$

for some $g : \mathbb{R}^{\mathrm{d}} \to (0,\infty)$, such that $g(x) \le BG(x)$, $x \in \mathbb{R}^{\mathrm{d}}$, with some $B > 0$. We consider properties of a general rate which satisfies (4) (e.g. the establishment rate has this property).

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Namely, we are going to find sufficient conditions for g such that

$$\langle g \rangle < 1 \tag{5}$$

would imply that the Dirac measure concentrated at $\varnothing$ is the only invariant distribution for η_t on Γ_G.

Theorem

Let (4) *hold with*

$$g(x) = b_g(|x|) \leq \frac{C}{(1+|x|)^{\mathrm{d}+2\varepsilon}}, \quad x \in \mathbb{R}^{\mathrm{d}},$$

for some $C > 0$*, where* $b_g : \mathbb{R}_+ \to (0, \infty)$ *is a continuously decreasing to* 0 *function. Suppose also that* (5) *holds. Then the Dirac measure concentrated at* $\emptyset$ *is the only invariant distribution for* η_t *on* Γ_G*.*

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Remark

In the case of the fecundity rate (2), $g(x) = r_p a(x)$, where

$$r_p := \sup_{s \in \mathbb{R}_+} (1+ps)e^{-s} = \begin{cases} 1, & 0 \le p \le 1, \\ pe^{\frac{1}{p}-1}, & p > 1, \end{cases}$$

i.e. (5) takes the form

$$r_p \langle a \rangle < 1.$$

Idea of proof

- Choose large $R > 0$, so that

$$\int_{|x|\geq R} (1+|x|)^{-d-\frac{3}{2}\varepsilon}\,dx < 1 - \langle g \rangle.$$

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- Choose large $R > 0$, so that

$$\int_{|x|\geq R} (1+|x|)^{-d-\frac{3}{2}\varepsilon}\, dx < 1 - \langle g \rangle.$$

- Choose a radially symmetric continuous bounded integrable function $c_g : \mathbb{R}^{\mathrm{d}} \to (0,\infty)$, so that

$$g(x) = b_g(|x|) \leq c_g(x) = \begin{cases} b_g(|x|), & |x| \leq R, \\ b_g(R), & R \leq |x| \leq R_1, \\ (1+|x|)^{-\mathrm{d}-\frac{3}{2}\varepsilon}, & |x| \geq R_1, \end{cases}$$

and $\langle c_g \rangle < 1$.

Idea of proof

- By using [F/Tkachov, *Advances in Applied Probability*'2018], show that
$$c_g^{*n}(x) \le C_{\alpha,\delta}\langle c_g\rangle^n (1+\delta)^n \min\Big\{\lambda_{\alpha,\delta},\, c_g(|x|)^\alpha\Big\}, \quad x \in \mathbb{R}^{\mathrm{d}},$$
for each small δ and for each $\alpha < 1$ close enough to 1.

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- Define
$$f(x) := c_g(x) + \sum_{n=2}^{\infty} c_g^{*n}(x), \quad x\in\mathbb{R}^{\mathrm{d}}$$
and show that, for $F(\eta)=\sum_{x\in\eta} f(x)$,
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$$LF(\eta) \le -\sum_{x\in\eta} c_g(x).$$
- Show that
$$F(\eta_t) + I_t, \quad \text{where } I_t := \int_0^t \sum_{x\in\eta_{s-}} c_g(x)\,ds,$$
is a non-negative supermartingale, and hence $\mathbb{E}\lim_{t\to\infty} I_t < \infty$.

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Thank you for your attention!