Doubly nonlocal Fisher–KPP equation: Features and peculiarities

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Introduction

- We will deal with the equation

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x-y)u(y,t)dy \\ - mu(x,t) - \varkappa^- u(x,t) \int_{\mathbb{R}^d} a^-(x-y)u(y,t)dy,\end{aligned} \qquad (*)$$

for $t>0$, $x \in \mathbb{R}^d$, $d \geq 1$, with an initial condition

$$u(x,0) = u_0(x), \qquad 0 \leq u_0 \in L^\infty(\mathbb{R}^d).$$

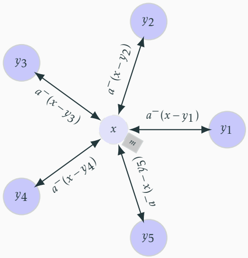
- Here $\varkappa^+, \varkappa^-, m > 0$ are constants, $a^+, a^- \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ are probability kernels:

$$a^\pm(x) \geq 0 \text{ a.e.}, \qquad \int_{\mathbb{R}^d} a^\pm(x)\,dx = 1.$$

This equation was derived from a model of mathematical ecology proposed by [Bolker/Pacala'97; Dieckmann/Law'00].

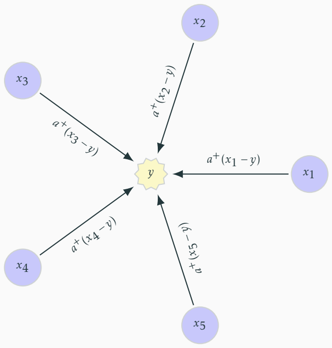
Let $\gamma = \gamma_t \subset \mathbb{R}^d$ denote a discrete set representing a population in $\mathbb{R}^d$ at a moment of time $t \geq 0$.

At a random moment of time, an existing element $x \in \gamma$ may disappear (die). The rate of this event depends on x itself, but also it is influenced by the rest of the population.



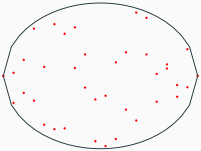
death rate $= m + \varkappa^- \sum_{y \in \gamma \setminus x} a^-(x-y)$

Also, at a random moment of time, an existing element x may send an off-spring to $y \in \mathbb{R}^d$. The rate of this event depends on both x and y only.

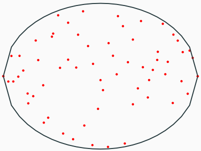


$$\text{birth rate in an area } \Lambda = \sum_{x \in \gamma} \varkappa^+ \int_\Lambda a^+(x-y)dy$$

Observe in a (small) region Λ



time $= 0$
initial (random) number of points $= N_0^\Lambda$



time $= t$
(random) number of points $= N_t^\Lambda$

Averaged over (thousands of) simulations:

$$\mathbb{E}\left[N_t^\Lambda\right] = \int_\Lambda k^{(1)}(x,t)\,dx,$$

$$\mathbb{E}\left[N_t^\Lambda(N_t^\Lambda - 1)\right] = \int_\Lambda \int_\Lambda k^{(2)}(x,y,t)\,dx\,dy,$$

$$\dots$$

The difficulty:

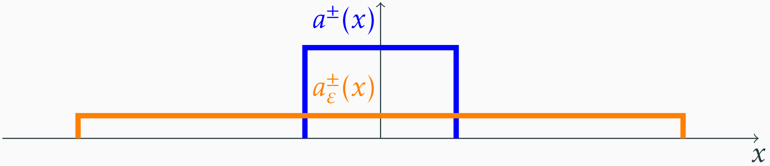
$$\frac{\partial}{\partial t} k^{(1)}(x,t) = \mathcal{L}\Big(k^{(1)}(\cdot,t), k^{(2)}(\cdot,\cdot,t)\Big)(x),$$

i.e. the equation is not closed.

(Mesoscopic) scaling:

$$a^{\pm}(x) \longmapsto a^{\pm}_{\varepsilon}(x) := \varepsilon^{d} a^{\pm}(\varepsilon x).$$

Note that $\int_{\mathbb{R}^d} a^{\pm}_{\varepsilon}(x)\,dx = 1.$



Let $N_{t,\varepsilon}^{\Lambda}$ will be the corresponding number of points in Λ at time t, and

$$\mathbb{E}\Big[N_{t,\varepsilon}^{\Lambda}\Big] = \int_{\Lambda} k_{\varepsilon}^{(1)}(x,t)\,dx.$$

Then

$$k_{\varepsilon}^{(1)}(x,t) = u(\varepsilon x, t) + O(\varepsilon^{d}),$$

where $u(x,t)$ solves $(*)$.

- Heuristically: [Bolker/Pacala'97] and [Dieckmann/Law'00]
- Rigorously for integrable u: [Fournier/Méléard'04]
- Rigorously for bounded u: [F/Kondratiev/Kutoviy'12], [F/Kondratiev/Kozitsky/Kutoviy'15]
- Equations for the the next term of the expansion of $k_\varepsilon^{(1)}$: [Ovaskainen/F/Kutoviy/Cornell/Bolker/Kondratiev'14]

Partial cases of the equation $(*)$:

- The case $a^+ = a^-$, $\varkappa^+ = \varkappa^-$, $m = 0$ was introduced by [Molisson'72]. 'Model of simple epidemics'.
- For $a^+ = a^-$, $\varkappa^+ = \varkappa^-$, $m > 0$, the equation was derived by [Durrett'88] from a 'crabgrass model' on $\mathbb{Z}^d$; viscosity solution method for the equation: [Pertham/Souganidis'05].

- We will always assume that

$$\varkappa^+ > m \tag{A1}$$

  to avoid that the solution degenerates for large times
- As a result, $u \equiv 0$ and

$$u \equiv \theta := \frac{\varkappa^+ - m}{\varkappa^-} > 0$$

  are stationary solutions to the equation $(*)$.

One can rewrite then $(*)$ in the reaction-diffusion form

$$\frac{\partial u}{\partial t}(x,t) = \varkappa^{+} \int_{\mathbb{R}^d} a^{+}(x-y)\Big(u(y,t) - u(x,t)\Big)dy \\ + \varkappa^{-} u(x,t)\bigg(\theta - \int_{\mathbb{R}^d} a^{-}(x-y)u(y,t)dy\bigg).$$

If $\int_{\mathbb{R}^d} x a^{+}(x)dx = 0 \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} |x|^2 a^{+}(x)dx < \infty$, then the formal scaling

$$\varkappa^{+} \mapsto \delta^{-2}\varkappa^{+}, \qquad a^{\pm}(x) \mapsto \delta^{-d} a^{\pm}\Big(\delta^{-1}x\Big)$$

leads to the classical Fisher–KPP equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha \Delta u(x,t) + \varkappa^{-} u(x,t)\Big(\theta - u(x,t)\Big),$$

for some $\alpha > 0$.

The equation

$$\frac{\partial u}{\partial t}(x,t) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x-y)\Big(u(y,t) - u(x,t)\Big)dy \\ + \varkappa^- u(x,t)\Big(\theta - u(x,t)\Big)$$

was studied in e.g.

[Bouin/Garnier/Henderson/Patout'18],

[Alfaro/Coville'17], [Berestycki/Coville/Vo'16],

[Bonnefon/Coville/Garnier/Roques'14],

[Coville/Dávila/Martínez'08,13],

[Aguerrea/Gomez/Trofimchuk'12], [Garnier'11],

[Li/Sun/Wang'10,11], [Yagisita'09],

[Coville/Dupaigne'05, 07], [Weng/Zhao'06],

[Hutson/Martinez/Mischaikow/Vickers'03],

[Schumacher'79, 80].

Comparison principle

Theorem (F/Tkachov'18 *Nonlinearity*)

Let $0 \leq u_0 \in L^\infty(\mathbb{R}^d)$. Then for any $T > 0$ there exists a unique classical nonnegative solution to $(*)$, i.e. such that the mapping $\mathbb{R}_+ \ni t \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d)$ is continuous on $[0, T]$ and continuously differentiable on $(0, T]$.

The classical comparison principle means, in particular, that

$$0 \le u(x,0) \le v(x,0)$$

leads to

$$0 \le u(x,t) \le v(x,t),$$

where $v(x,t)$ is the corresponding solution to $(*)$ with the initial condition $v(x,0)$.

A peculiarity of the nonlocal reaction is the following:

Theorem (F/Tkachov'18 *Nonlinearity*; F/Kondratiev/Tkachov'19 *Electr. J. Diff. Eqns.*)

Let (A1) *hold. The comparison principle for* $(*)$ *holds if and only if, for a.a.* $x \in \mathbb{R}^d$,

$$\varkappa^+ a^+(x) \ge (\varkappa^+ - m) a^-(x). \tag{A2}$$

Corollary

Let (A1)–(A2) hold. Suppose that $0 \le u_0(x) \le \theta$ *for a.a.* $x \in \mathbb{R}^d$. *Then, for all* $t > 0$ *and a.a.* $x \in \mathbb{R}^d$,

$$0 \le u(x,t) \le \theta.$$

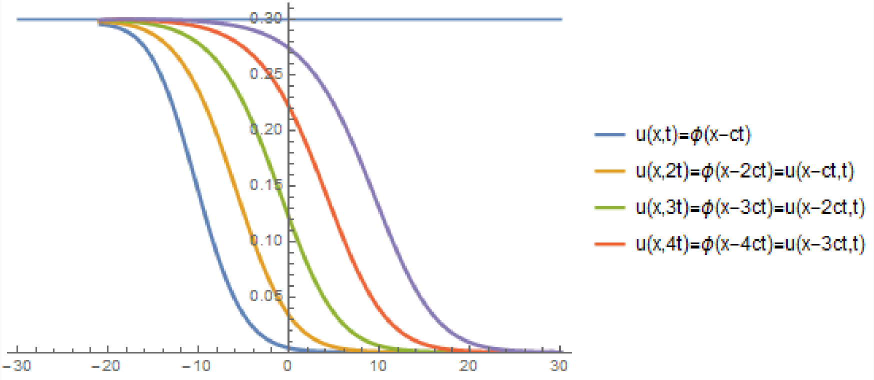
Travelling waves

Definition

We will say that a solution u to $(*)$ is a (monotone) travelling wave solution with a speed $c \in \mathbb{R}$ and in a direction $\xi \in S^{d-1}$ if there exists a decreasing and right-continuous function (the profile) $\psi : \mathbb{R} \to [0, \theta]$, such that

$$u(x,t) = \psi(x \cdot \xi - ct), \quad t \geq 0, \text{ a.a. } x \in \mathbb{R}^d,$$

$$\psi(-\infty) = \theta, \qquad \psi(+\infty) = 0,$$



Known results: [Weng/Zhao'06]: $d = 1$, symmetric $a^+ = a^-$ which decays faster than any exponential function; [Yu/Yuan'13]: the same but for different symmetric a^+ and a^-.

We assume that, for a fixed direction $\xi \in S^{d-1}$,

$$\begin{aligned} \mathfrak{a}_\xi(\nu) &:= \int_{\mathbb{R}^d} a^+(x) e^{\nu x\cdot\xi}\, dx < \infty \qquad \text{for **some** } \nu = \nu(\xi) > 0, \\ \|u_0\|_{\lambda,\xi} &:= \operatorname*{esssup}_{x\in\mathbb{R}^d} u_0(x) e^{\lambda x\cdot\xi} < \infty \quad \text{for **all** } \lambda > 0. \end{aligned} \tag{A3}$$

Theorem (F/Kondratiev/Tkachov'19 *Electr. J. Diff. Eqs. & J. Math. Anal. Appl.*)

Let (A1)–(A3) hold. Then there exists

$$c_\xi = \min_{\mu>0} \frac{\varkappa \mathfrak{a}_\xi(\mu) - m}{\mu} \in \mathbb{R},$$

*such that, for any $c \geq c_\xi$, there exists a **unique** (up to shifts) monotone travelling wave, and for $c < c_\xi$ such waves do not exist. In the former case the profile is continuous (and even smooth unless $c = 0$) and strictly monotone.*

Theorem (F/Kondratiev/Tkachov'19 *J. Math. Anal. Appl.*)

Let (A1)–(A3) hold. For any $c \geq c_\xi$, there exists $\sigma_c \in (0,\infty)$ such that the asymptotic of the corresponding monotone profile ψ_c at $+\infty$ is given by

$$\psi_c(s) \sim D\, s^j\, e^{-\sigma_c s}, \quad s \to \infty.$$

Here $j = 0$ if either $c > c_\xi$ or $c = c_\xi$ and

$$\widehat{\sigma} := \sup_{\lambda>0}\{\mathfrak{a}(\lambda) < \infty\} < \infty,$$

$$\int_{\mathbb{R}} (1+|x|)a^+(x)e^{\widehat{\sigma}x}\,dx < \infty,$$

$$m \leq \widehat{m} := \varkappa^+ \int_{\mathbb{R}} (1-\widehat{\sigma}x)a^+(x)e^{\widehat{\sigma}x}\,dx.$$

Otherwise $j = 1$.

Example

Let $d = 1$. Consider

$$a^+(x) := \frac{\alpha e^{-\mu|x|}}{1+|x|^q}, \qquad x \in \mathbb{R},\ q \geq 0,\ \mu > 0,$$

where $\alpha > 0$ is a normalizing constant. Then $\widehat{\sigma} = \mu$ and, for $q > 2$, there exist $\mu_* > 0$ and $m_* \in (0, \varkappa^+)$, such that $j = 0$ for $c = c_\xi$ provided that $\mu \in (0, \mu_*]$ and $m \in (0, m_*]$.

The technique is based on a Tauberian-Ikehara-type theorem, see [F/Tkachov' 19 *Comptes Rendus Mathématique*].

Front propagation

Theorem (F/Kondratiev/Tkachov'19, Applicable Analysis *(accepted)*)

Let (A1)–(A3). Let μ_ξ be such that $c_\xi = \frac{\varkappa \mathfrak{a}_\xi(\mu_\xi)-m}{\mu_\xi}$. Then

$$0 \le u(x,t) \le \|u_0\|_{\mu_\xi,\xi}\, e^{\mu_\xi(c_\xi t - x\cdot\xi)}.$$

Moreover, consider

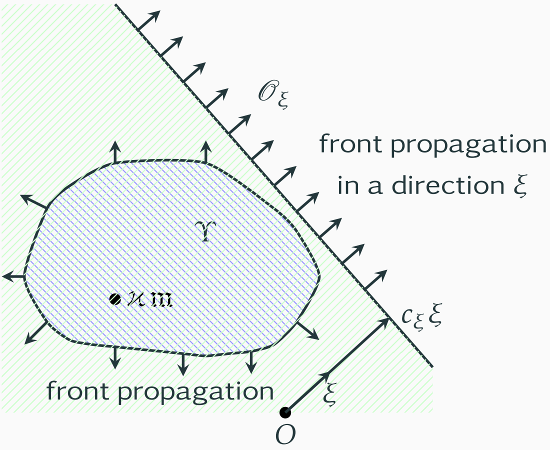
$$\Upsilon := \Bigl\{x\in\mathbb{R}^d \Bigm| x\cdot\xi \le c_\xi,\ \xi\in S^{d-1}\Bigr\}.$$

Then, for any open $\mathscr{T}\supset\Upsilon$, there exist $\nu, C>0$, such that

$$0 < u(x) \le C\, e^{-\nu t}, \qquad x\in t\mathscr{T}^c,\ t>0.$$

Finally, if (A3) holds for all $\xi\in S^{d-1}$, then, for any compact $\mathscr{C}\subset \operatorname{int}(\Upsilon)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\in t\mathscr{C}} u(x,t) = \theta.$$



For the case $\varkappa^+ = \varkappa^-$, $a^+ = a^-$ see also [Perthame/Souganidis'05].

- Consider the case $d = 1$.
- We will distinguish two cases for the initial condition $u_0 : \mathbb{R} \to \mathbb{R}_+$:

$$\lim_{x \to \pm\infty} u_0(x) = 0, \tag{C1}$$

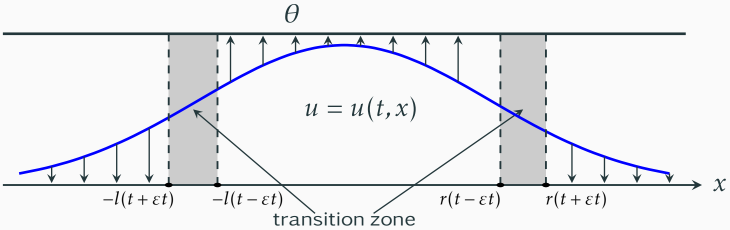
$$\lim_{x \to \infty} u_0(x) = 0, \qquad \inf_{x \leq -\rho} u_0(x) > 0 \tag{C2}$$

  for some $\rho \geq 0$.
- Let $r, l : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing to ∞ functions, such that the following holds.

Case (C1) For each $\varepsilon \in (0, 1)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{[-l(t-\varepsilon t), r(t-\varepsilon t)]} u(x,t) = \theta.$$

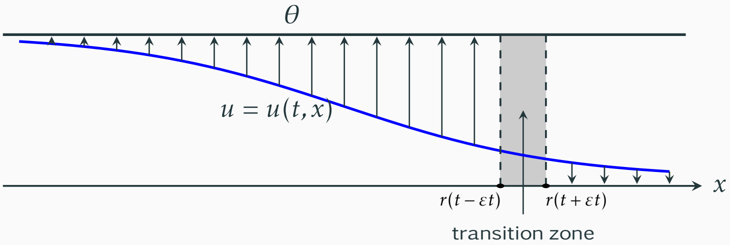
$$\lim_{t\to\infty} \operatorname*{esssup}_{(-\infty,-l(t+\varepsilon t)]\cup[r(t+\varepsilon t),\infty)} u(x,t) = 0.$$



Case (C2) For each $\varepsilon \in (0,1)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{(-\infty, r(t-\varepsilon t)]} u(x,t) = \theta.$$

$$\lim_{t\to\infty} \operatorname*{esssup}_{[r(t+\varepsilon t), \infty)} u(x,t) = 0.$$



- Finite speed of propagation (to the right):

$$\begin{aligned} r(t) &= c_{\xi=1} t, \\ l(t) &= c_{\xi=-1} t, \end{aligned} \qquad \text{note that } \Upsilon = [-c_{\xi=-1}, c_{\xi=1}].$$

- Acceleration (to the right):

$$\lim_{t\to\infty} \frac{r(t)}{t} = \infty.$$

Theorem (**Informal. Full version:** F/Tkachov'19 *Applicable Analysis*)

- *Acceleration takes place if* ***either*** a ***or*** u_0 *(or both) have heavy tails.*
- *For the case (C1),* $r(t)$ *is described by the heaviest right tail (slowest decaying at* $+\infty$*) of* a^+ *and* u_0 *.*
- *For the case (C1),* $l(t)$ *is described (independently) by the heaviest left tail (slowest decaying at* $-\infty$*) of* a^+ *and* u_0*.*
- *For the case (C2), to describe* $r(t)$*, we have to consider* $\int_x^\infty a^+$ *and* u_0 *instead.*

Let $b(x)$ denote a function with the heaviest tail as described above. Consider the explicit form of $r(t)$.

$$b(x) = (\log x)^{\mu} x^{-q}, \qquad r(t) = \exp\left(\frac{\beta}{q} t\right);$$

$$b(x) = (\log x)^{\mu} x^{\nu} \exp\left(-p(\log x)^{q}\right), \qquad r(t) = \exp\left(\left(\frac{\beta}{p} t\right)^{\frac{1}{q}}\right);$$

$$b(x) = (\log x)^{\mu} x^{\nu} \exp\left(-x^{\alpha}\right), \qquad r(t) = (\beta t)^{\frac{1}{\alpha}};$$

$$b(x) = (\log x)^{\mu} x^{\nu} \exp\left(-\frac{x}{(\log x)^{q}}\right), \qquad r(t) \sim \beta t (\log t)^{q}, t \to \infty.$$

Here $\mu, \nu \in \mathbb{R}$, $q > 1$, $\alpha \in (0,1)$, $p > 0$.

Let $n \neq 0$, $r > 1$, $\delta > 0$, $\alpha \in (0,1)$. Suppose that

$$\text{for } x > r \qquad \begin{cases} \dfrac{1}{(1+x)^n}\exp(-x^\alpha) \leq a^+(x) \leq (1+x)^n \exp(-x^\alpha), \\ u_0(x) \leq (1+x)^n \exp(-x^\alpha), \end{cases} ,$$

$$\text{for } x < -r \quad \begin{cases} \dfrac{1}{(\log(-x))^n}\dfrac{1}{(1-x)^{1+\delta}} \leq u_0(x) \leq (\log(-x))^n \dfrac{1}{(1-x)^{1+\delta}}, \\ a^+(x) \leq (\log(-x))^n \frac{1}{(1-x)^{1+\delta}}, \end{cases}$$

Then

$$\lim_{t\to\infty} \operatorname*{essinf}_{\Lambda(t-\varepsilon t)} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{esssup}_{\mathbb{R}\setminus\Lambda(t+\varepsilon t)} u(x,t) = 0,$$

where

$$\Lambda(t) = \left[-\exp\left(\frac{\beta t}{1+\delta}\right), (\beta t)^{\frac{1}{\alpha}}\right].$$

Our technique is based on properties of n-fold convolutions for heavy-tailed functions, see [F/Tkachov'18 Adv.Appl.Prob.]

Other known results:

- For $d > 1$, both local and nonlocal reactions [F/Kondratiev/Tkachov'18, J. Elliptic & Parabolic Eqns. (accepted)].
- For $d = 1$ and local reaction:
  - [Garnier'11]: compactly supported u_0, symmetric a^+, convergence to 0 was shown on $\{|x| \geq r(\gamma t)\}$ with an unknown $\gamma > 1$ (instead of $\gamma = 1 + \varepsilon$ for an arbitrary ε);
  - [Bouin/Garnier/Henderson/Patout'18]: symmetric and heavy-tailed $a \geq c u_0$.

- Let $u_0(x)$ decays faster at $+\infty$ than $a^+(x)$ which has a regularly heavy tail at $+\infty$. Then the propagation to the right will be characterized by $a^+(x)$ if $u_0(x)$ tends to 0 at $-\infty$ and by $\int_x^\infty a^+(y)\,dy$ if $u_0(x)$ is separated from 0 at $-\infty$.
- If
$$\log a^+(x) = o\Bigl(\log \int_x^\infty a^+(y)\,dy\Bigr), \quad x \to \infty,$$
then we will get different $r(t)$ in (C1) and (C2).
- This is the case, e.g., for
$$a^+(x) = x^{-q} \quad \text{and} \quad a^+(x) = \exp\Bigl(-p(\log x)^q\Bigr).$$

- Similar results were obtained for the equation with fractional Laplacian instead of $\varkappa a^+ * u - \varkappa u$, i.e., informally, for

$$a^+(x) = \frac{1}{|x|^{d+2\delta}}, \quad \delta \in (0,1),$$

see [Cabré/Roquejoffre'13], [Felmer/Yangari'13], [Méléard/Mirrahimi'15].

- However, their approaches don't yield a result for

$$a^+(x) = \frac{1}{1+|x|^{d+2\delta}}, \quad \delta \in (0,1).$$

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Thank you for your attention!