n-fold Convolutions of Probability Densities with Regular Heavy Tails

Based on joint papers with Pasha Tkachov (L'Aquila, Italy)

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Dmitri Finkelshtein

Swansea University

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Introduction

Tail functions

- For a probability distribution (probability measure) F on $\mathbb{R}$, let

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$$\overline{F_1 * F_2}(s) = \int_{\mathbb{R}} \overline{F}_1(s-\tau)F_2(d\tau) = \int_{\mathbb{R}} \overline{F}_2(s-\tau)F_1(d\tau).$$

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- Recall that $X_1 \sim F_1$, $X_2 \sim F_2$ implies $X_1 + X_2 \sim F_1 * F_2$.

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- Recall that $X_1 \sim F_1, X_2 \sim F_2$ implies $X_1 + X_2 \sim F_1 * F_2$.
- Let F be concentrated on $\mathbb{R}_+ := [0,\infty)$ and $\overline{F}(s) > 0$, $s \in \mathbb{R}$, then

$$\liminf_{s\to\infty} \frac{\overline{F * F}(s)}{\overline{F}(s)} \geq 2.$$

Chistyakov'1964

- Let, additionally, F be *heavy-tailed*, i.e.

$$\int_{\mathbb{R}} e^{\lambda s} F(ds) = \infty \qquad \text{for all } \lambda > 0,$$

then the equality holds:

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Foss/Korshunov'2007

- *Definition.* A distribution F concentrated on $\mathbb{R}_+$ is said to be *sub-exponential*, if

$$\lim_{s\to\infty} \frac{\overline{F * F}(s)}{\overline{F}(s)} = 2.$$

Chistyakov'1964; Hover/Ney/Wainger'1969; Athreya/Ney'1972

Chistyakov'1964 has also shown that:

- Any sub-exponential distribution is *long-tailed*, i.e.

$$\lim_{s\to\infty} \frac{\overline{F}(s+t)}{\overline{F}(s)} = 1 \quad \text{for each } t > 0.$$

(Moreover, each long-tailed distribution is heavy-tailed.)

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- If F is a sub-exponential distribution concentrated on $\mathbb{R}_+$, then

$$\lim_{s\to\infty} \frac{\overline{F^{*n}}(s)}{\overline{F}(s)} = n,$$

  where $F^{*n} := F * \ldots * F$ $(n-1$ times).

Sub-exponential distributions: properties

- If $X_1 \geq 0, \ldots, X_n \geq 0$ are i.i.d.r.v. with a sub-exponential distribution, then

$$\mathbb{P}\Big(X_1 + \ldots + X_n > s\Big) \sim \mathbb{P}\Big(\max\{X_1, \ldots, X_n\} > s\Big), \quad s \to \infty.$$

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- Were used by `Chistyakov'`1964 and later by `Athreya/Ney'`1972 for the study of the renewal equation and branching processes. For this (and later for risk theory) one needs 'more uniform' in $n \in \mathbb{N}$ bound instead of

$$\overline{F^{*n}}(s) \leq (n + \delta)\overline{F}(s), \qquad s > s_\delta(n).$$

Kesten's bound for distributions on $\mathbb{R}_+$

- Let F be a sub-exponential distribution concentrated on $\mathbb{R}_+$, then, for each $\delta > 0$, there exists $c_\delta > 0$, such that

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- History: Chistyakov'1964: under additional assumptions, general case: Athreya/Ney'1972 (the proof was proposed by Kesten). We follow the terminology by Foss/Korshunov/Zachary'2013.
- The 'profit': uniform convergence of series

$$\sum_{n=1}^{\infty} \lambda_n \overline{F^{*n}}.$$

Were used in branching age dependent processes, random walks, queue theory, risk theory and ruin probabilities, compound Poisson processes, and the study of infinitely divisible laws.

- If distributions F_1, F_2 on $\mathbb{R}$ have probability densities $b_1 \geq 0, b_2 \geq 0$, with $\int_{\mathbb{R}} b_1(s)\,ds = \int_{\mathbb{R}} b_2(s)\,ds = 1$, then $F_1 * F_2$ has the density

$$(b_1 * b_2)(s) := \int_{\mathbb{R}} b_1(s-t) b_2(t)\,dt, \quad s \in \mathbb{R}.$$

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- The density b of a sub-exponential distribution F concentrated on $\mathbb{R}_+$ (i.e. $b(s) = 0$ for $s < 0$) is said to be sub-exponential on $\mathbb{R}_+$ if b is long-tailed, i.e.

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- Note that any long-tailed function b satisfies

$$\lim_{s\to\infty} e^{\lambda s} b(s) = \infty \qquad \text{for each } \lambda > 0.$$

Sub-exponential densities on $\mathbb{R}_+$: properties

- Let b be a sub-exponential density on $\mathbb{R}_+$ (recall that $b(s) = 0$ for $s < 0$). Then

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- Moreover, the following Kesten's bound hold: for any $\delta > 0$, there exist $s_\delta > 0$ and $c_\delta > 0$, such that

$$b^{*n}(s) \leq c_\delta (1+\delta)^n b(s), \quad s \geq s_\delta,\ n \in \mathbb{N}.$$

Klüppelberg'1989; Asmussen/Foss/Korshunov'2003

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- Note that if b is a sub-exponential density on $\mathbb{R}_+$, then F is a sub-exponential distribution on $\mathbb{R}_+$, but the converse result is not, in general, true.

The following functions, being normalized on $\mathbb{R}_+$, become sub-exponential densities of

- Student's t-distribution.

$$b(s) = \frac{1}{\left(1 + \frac{s^2}{2p-1}\right)^p}, \qquad p > \frac{1}{2}.$$

$p = 1$ corresponds to the Cauchy distribution

- The Lévy distribution

$$b(s) = (s-\mu)^{-\frac{3}{2}} \exp\left(-\frac{c}{s-\mu}\right), \qquad c > 0,\ \mu \in \mathbb{R}.$$

- The Burr IV distribution.

$$b(s) = \frac{s^{c-1}}{(1+s^c)^{k+1}}, \qquad c > 0, k > 0.$$

$c = 1$ is related to the Pareto distribution.

The following functions, being normalized on $\mathbb{R}_+$, become sub-exponential densities of

- The log-normal distribution.

$$b(s) = \frac{1}{s}\exp\left(-\frac{(\log s - \mu)^2}{2\gamma^2}\right), \qquad \gamma > 0,\ \mu \in \mathbb{R}.$$

- The Weibull distribution.

$$b(s) = \frac{\exp(-s^\alpha)}{s^{1-\alpha}}, \qquad \alpha \in (0,1).$$

- 'Almost exponential' distribution.

$$b(s) = \exp\left(-\frac{s}{(\log s)^\alpha}\right), \qquad \alpha > 0.$$

Sub-exponential densities and Kesten's bound on $\mathbb{R}$

Sub-exponential distributions on the whole $\mathbb{R}$

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- Therefore, a general distribution on $\mathbb{R}$ (with right-unbounded support) is said to be sub-exponential if $\overline{F * F}(s) \sim 2\overline{F}(s)$, $s \to \infty$ **and** $\overline{F}$ is long-tailed that is, recall,

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- Then $\overline{F^{*n}}(s)\sim n\overline{F}(s)$, $s\to\infty$ for any $n\geq 2$ and Kesten's bound remains unchanged.

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- For more deep properties and differences with the $\mathbb{R}_+$ case see Watanabe'2008.

Sub-exponential densities on the whole $\mathbb{R}$: definition

- We will say that a density b is (right-side) sub-exponential on $\mathbb{R}$ if b is (right-side) long-tailed, i.e.

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- Asmussen/Foss/Korshunov'2003 have shown that if a density b on $\mathbb{R}$ is long-tailed and, being restricted and normalized on $\mathbb{R}_+$, becomes a sub-exponential density on $\mathbb{R}_+$, and if, additionally, the condition

$$b(s+\tau) \le Kb(s), \quad s > \rho,\ \tau > 0 \tag{1}$$

holds for some $K > 0$ and $\rho > 0$, then b is a sub-exponential density on $\mathbb{R}$.

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holds for some $K > 0$ and $\rho > 0$, then b is a sub-exponential density on $\mathbb{R}$.
- In particular, if b is *tail-decreasing*, i.e. decays to 0 on $[\rho,\infty)$ for some $\rho > 0$, then (1) holds.

Definition

For a (probability) density b on $\mathbb{R}$, let b_+ denote its normalized restriction to $\mathbb{R}_+$.

Sub-exponential densities on the whole $\mathbb{R}$: result

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For a (probability) density b on $\mathbb{R}$, let b_+ denote its normalized restriction to $\mathbb{R}_+$.

Theorem 1

Let b be a density on $\mathbb{R}$, such that b_+ is a sub-exponential density on $\mathbb{R}_+$, and let (1) holds (for example, let b be tail-decreasing). Then

$$b^{*n}(s) \sim nb(s), \quad s \to \infty,\ n \geq 2.$$

Sub-exponential densities on the whole $\mathbb{R}$: background

The proof follows from

Proposition 1

Let $b : \mathbb{R} \to \mathbb{R}_+$ satisfy the conditions above. Let $b_1, b_2 \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and there exist constants $c_1, c_2 \geq 0$, such that

$$\lim_{s\to\infty} \frac{b_j(s)}{b(s)} = c_j, \quad j = 1, 2.$$

Then

$$\lim_{s\to\infty} \frac{(b_1 * b_2)(s)}{b(s)} = c_1 \int_{\mathbb{R}} b_2(\tau)\, d\tau + c_2 \int_{\mathbb{R}} b_1(\tau)\, d\tau.$$

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Then, in particular, $b_1 = b$, $b_2 = o(b)$ imply $b * b_2 \sim b$.

Theorem 2

Let b be a bounded density on $\mathbb{R}$, such that b_+ is a sub-exponential density on $\mathbb{R}_+$, and let (1) holds (e.g., let b be tail-decreasing). Then, for any $\delta \in (0,1)$, there exist $C_\delta > 0$ and $s_\delta > 0$, such that

$$b^{*n}(s) \le C_\delta (1+\delta)^n b(s), \qquad s > s_\delta,\ n \in \mathbb{N}.$$

Application to the non-local heat equation on $\mathbb{R}$

Consider the non-local heat equation on $\mathbb{R}$

$$\frac{\partial}{\partial t}u(x,t) = \varkappa \int_{\mathbb{R}} a(x-y)\Big(u(y,t) - u(x,t)\Big)\,dy, \quad x \in \mathbb{R},$$

where $\varkappa > 0$ and $0 \le a \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} a(x)\,dx = 1$. Let $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, where $0 \le u_0 \in L^\infty(\mathbb{R})$.

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The unique solution in $L^\infty(\mathbb{R})$ is

$$u(x,t) = e^{-\varkappa t}u_0(x) + e^{-\varkappa t}\Big(\phi_\varkappa(t) * u_0\Big)(x),$$

where

$$\phi_\varkappa(x,t) := \sum_{n=1}^{\infty} \frac{\varkappa^n t^n}{n!} a^{*n}(x), \quad x \in \mathbb{R},\ t \ge 0.$$

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If a satisfies the conditions of Theorem 2, then the series above converges uniformly on finite time intervals for each $x > s_\delta$, and therefore, by Theorem 1,

$$\phi_\varkappa(x,t) \sim \varkappa t e^{\varkappa t} a(x), \quad x \to \infty,\ t > 0.$$

Sub-exponential densities on the whole $\mathbb{R}$: technical tools

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- *Definition.* b is said to be h-insensitive w.r.t. an increasing function h, such that $0 < h(s) < \frac{s}{2}$ and $h(s) \to \infty$, $s \to \infty$, if

$$\sup_{|t|\leq h(s)} \left| \frac{b(s+t)}{b(s)} - 1 \right| = 0. \qquad (2)$$

- h-insensitive property: proposed by Asmussen/Foss/Korshunov'2003.
- If b is long-tailed, then the convergence $\frac{b(s+t)}{b(s)} \to 1, s \to \infty$ is locally uniform in t: for each $h > 0$,

$$\sup_{|t|\le h}\left|\frac{b(s+t)}{b(s)} - 1\right| = 0.$$

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- For each long-tailed b such h does exist (not unique, of course).

Sub-exponential densities: sufficient conditions

- Asmussen/Foss/Korshunov'2003 have shown that if b is long-tailed and *tail-log-convex*, i.e. $\log b$ is convex on (ρ, ∞) for some $\rho > 0$, and the function h above is such that

$$\lim_{s \to \infty} s\, b\big(h(s)\big) = 0, \tag{3}$$

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- Hence, if, additionally, (1) holds (e.g. if b is tail-decreasing), then, by Theorem 1, b is sub-exponential on $\mathbb{R}$ and, by Theorem 2, Kesten's bound holds.

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- We denote by $\mathcal{S}_0$ the class of 'regular' densities which are tail-decreasing, tail-log-convex, and there exists h as the above (i.e. $\frac{s}{2}>h(s)\nearrow\infty$), such that (3)–(4) hold.

Sub-class of regular densities on $\mathbb{R}_+$

- For any $b \in \mathcal{S}_0$, both Theorems 1 and 2 hold. It is natural to find transformations which keep functions in $\mathcal{S}_0$ or, at least, in some its subclasses.

Sub-class of regular densities on $\mathbb{R}_+$

- For any $b \in \mathcal{S}_0$, both Theorems 1 and 2 hold. It is natural to find transformations which keep functions in $\mathcal{S}_0$ or, at least, in some its subclasses.
- Consider a sub-class $\mathcal{S}_d$, $d \geq 1$ of the class $\mathcal{S}_0$ of regular densities b on $\mathbb{R}$, such that $b \in L^1(\mathbb{R}_+, s^{d-1}\,ds)$, and, for some $\delta = \delta(b) > 0$ and h as above,

$$\lim_{s\to\infty} s^{1+\delta} b\Big(h(s)\Big) = 0. \tag{5}$$

Reason 1 for the sub-class

Definition. The densities b and c, positive 'at infinity', are said to be *log-equivalent* if

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Proposition 2

Let $b \in \mathcal{S}_d$ and let h be the corresponding function. Let $c : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex density, such that

$$\lim_{s\to\infty} \frac{c(s \pm h(s))}{c(s)} = 1.$$

Suppose that b and c are log-equivalent. Let also, for some $\alpha \in (0,1)$, $b^\alpha \in L^1(\mathbb{R}_+, s^{d-1}\,ds)$. Then $c \in \mathcal{S}_d$.

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Typical application: $c(s) = p(s)b(s)$, $s \in \mathbb{R}_+$ with $\log p = o(\log b)$.

(New) examples of sub-exponential densities

Let $b : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex density, such that, for some $C > 0$, $\nu, \mu \in \mathbb{R}$, the function $C\, b(s)$ has either of the following asymptotics as $s \to \infty$

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- $(\log s)^\mu s^{-(d+\delta)}$,
- $(\log s)^\mu s^\nu \exp\Big(-D(\log s)^q\Big)$,
- $(\log s)^\mu s^\nu \exp\Big(-s^\alpha\Big)$,
- $(\log s)^\mu s^\nu \exp\Big(-\frac{s}{(\log s)^q}\Big)$,

where $D, \delta > 0$, $q > 1$, $\alpha \in (0,1)$. Then $b \in \mathcal{S}_d$, $d \geq 1$.

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- $(\log s)^\mu s^{-(d+\delta)}$, $\qquad h(s) = s^\beta,\ \beta \in \left(\frac{1}{d+\delta}, 1\right)$;
- $(\log s)^\mu s^\nu \exp\left(-D(\log s)^q\right)$, $\qquad h(s) = s^{\frac{1}{q}}$;
- $(\log s)^\mu s^\nu \exp\left(-s^\alpha\right)$, $\qquad h(s) = (\log s)^{\frac{2}{\alpha}} < s^\beta$;
- $(\log s)^\mu s^\nu \exp\left(-\frac{s}{(\log s)^q}\right)$, $\qquad h(s) = (\log s)^\beta,\ \beta \in (1, q)$,

where $D, \delta > 0$, $q > 1$, $\alpha \in (0,1)$. Then $b \in \mathcal{S}_d$, $d \geq 1$.

Proposition 3

Let $b \in \mathcal{S}_d$ and, for some $\alpha_0 \in (0,1)$, $b^{\alpha_0} \in L^1(\mathbb{R}_+, s^{d-1}\,ds)$. Then there exists $\alpha_1 \in (\alpha_0, 1)$, such that, for all $\alpha \in [\alpha_1, 1]$,

$$b^\alpha \in \mathcal{S}_d.$$

Kesten-type bound on $\mathbb{R}^d$

Challenge

- The multi-dimensional version of the constructions above is much more non-trivial.

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- The variety is mainly related to different possibilities to describe the zones in $\mathbb{R}^d$ where an analogue of the equivalence $\overline{F * F} \sim 2\overline{F}$ takes place.
- Any results about sub-exponential densities in $\mathbb{R}^d$, $d > 1$, seem to be absent at all.
- Note that properties of the distribution tails and the integrated tails of the corresponding densities are not related in the multi-dimensional case, since, for a probability density a on $\mathbb{R}^d$,

$$1 - \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} a(y)\, dy \neq \int_{x_1}^{\infty} \dots \int_{x_d}^{\infty} a(y)\, dy,$$

unless $d = 1$.

Explanation

- Note also that if, e.g. a is radially symmetric, i.e. $a(x) = b(|x|)$, $x \in \mathbb{R}^d$ (here $|x|$ denotes the Euclidean norm on $\mathbb{R}^d$) and b, being normalized, is a sub-exponential density on $\mathbb{R}_+$, then

$$(a * a)(x) := \int_{\mathbb{R}^d} a(x-y)a(y)\,dy = c(|x|), \quad x \in \mathbb{R}^d,$$

for some $c : \mathbb{R}_+ \to \mathbb{R}_+$, i.e. $a * a$ is also radially symmetric, however, asymptotic behaviors of b and c at ∞ are hardly to be compared.

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or

$$\lim_{s\to\infty} b(s)s^\nu = 0, \qquad \text{for all } \nu \geq 1.$$

An analogue of Kesten's bound on $\mathbb{R}^d$

Theorem 3

1. *Let* $a(x) = b(|x|)$, $x \in \mathbb{R}^d$ *for some* $b \in \tilde{\mathcal{S}}_d$, $d \geq 1$. *Then there exists* $\alpha_0 \in (0,1)$, *such that, for any* $\delta \in (0,1)$ *and* $\alpha \in (\alpha_0, 1)$, *there exist* $c_{\delta,\alpha} > 0$ *and* $s_{\delta,\alpha} > 0$, *such that*

$$a^{*n}(x) \leq c_{\delta,\alpha}(1+\delta)^n a(x)^\alpha, \quad |x| \geq s_{\delta,\alpha},\ n \in \mathbb{N}.$$

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2. *Let* $a(x) \leq c(|x|)$, $x \in \mathbb{R}^d$, *such that* $\log c(s) \sim \log b(s)$, $s \to \infty$ *for some* $b \in \tilde{\mathcal{S}}_d$, $d \geq 1$. *Then there exists* $\alpha_0 \in (0,1)$, *such that, for any* $\delta \in (0,1)$ *and* $\alpha \in (\alpha_0, 1)$, *there exist* $c_{\delta,\alpha} > 0$ *and* $s_{\delta,\alpha} > 0$, *such that*

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Application to the non-local heat equation in $\mathbb{R}^d$

Consider now the non-local heat equation in $\mathbb{R}^d$

$$\frac{\partial}{\partial t}u(x,t) = \varkappa \int_{\mathbb{R}^d} a(x-y)\Big(u(y,t)-u(x,t)\Big)dy, \quad x \in \mathbb{R}^d,$$

where $\varkappa > 0$ and $0 \le a \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} a(x)\,dx = 1$. Let $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$, where $0 \le u_0 \in L^\infty(\mathbb{R}^d)$.

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$$u(x,t) = e^{-\varkappa t}u_0(x) + e^{-\varkappa t}\Big(\phi_\varkappa(t) * u_0\Big)(x),$$

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$$\phi_\varkappa(x,t) := \sum_{n=1}^{\infty} \frac{\varkappa^n t^n}{n!} a^{*n}(x), \quad x \in \mathbb{R}^d,\ t \ge 0.$$

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Then, under the conditions of Theorem 3,

$$\phi_\varkappa(x,t) \le c_{\delta,\alpha}\Big(e^{\varkappa t(1+\delta)} - 1\Big)b(|x|)^\alpha, \quad |x| > s_{\delta,\alpha},\ t > 0$$

for each $\alpha < 1$ close enough to 1.

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Thank you for your attention!